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# INVOLUTIONS ON $S^1 \times S^2$ AND OTHER 3-MANIFOLDS

BY

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ABSTRACT. This paper exploits the following observation concerning involutions on nonirreducible 3-manifolds: If the dimension of the fixed point set of a PL involution is less than or equal to one then there exists a pair of disjoint 2-spheres that do not bound 3-cells and whose union is invariant under the given involution. The classification of all PL involutions of $S^1 \times S^2$ is obtained. In particular, $S^1 \times S^2$ admits exactly thirteen distinct PL involutions (up to conjugation). It follows that there is a unique PL involution of the solid torus $S^1 \times D^2$ with 1-dimensional fixed point set. Furthermore, there are just four fixed point free $Z_{2k}$-actions and just one fixed point free $Z_{2k+1}$-action on $S^1 \times S^2$ for each positive integer $k$ (again, up to conjugation). The above observation is also used to obtain a general description of compact, irreducible 3-manifolds that admit two-sided embeddings of the projective plane.

1. **Introduction.** We have two main goals in this paper. The first is to complete the classification of PL involutions on $S^1 \times S^2$. As we show, $S^1 \times S^2$ admits only the obvious involutions. Secondly, we give a characterization of compact, irreducible 3-manifolds admitting two-sided embeddings of the projective plane. An elementary observation concerning involutions on nonirreducible 3-manifolds (Lemma 1) inspired both solutions.

Let $T$ be a PL involution of $S^1 \times S^2$ with fixed point set $F$. In [11], Tao has proven that if $F = \emptyset$ then the orbit space of $T$ is homeomorphic to $S^1 \times S^2$, $P^2 \times S^1$, $P^3 \# P^3$, or $N$ ($N$ denotes the nonorientable 2-sphere bundle over $S^1$). Kwun [5] and Fremon [2] have shown that if $F$ is 2-dimensional, then $T$ is uniquely determined (up to equivalence) by its fixed point set. Hence there are only four involutions $T$ with 2-dimensional fixed point set. In this paper we treat the remaining cases, namely, when $F$ has dimension 0 or 1. We show that there is only one involution when the dimension of $F$ is 0 and that there are just four involutions when the dimension of $F$ is 1.

We describe the standard involutions on $S^1 \times S^2$ with fixed point sets homeomorphic to $S^0 \cup S^0$, $S^1$, and $S^1 \cup S^1$. Let $C: S^1 \to S^1$ denote the reflection with two fixed points. Regard $S^2$ as the unit sphere $\{(x, y, z): x^2 + y^2 + z^2 = 1\}$

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in $R^3$. Let the involutions $A$, $K$, $R_1$, $R_2$ of $S^2$ be defined as follows:

$$K(x, y, z) = (x, y, -z), \qquad R_1(x, y, x) = (-x, -y, z),$$
$$A(x, y, z) = (-x, -y, -z), \qquad R_2(x, y, z) = (x, -y, -z).$$

$C \times R_1$ is the standard involution on $S^1 \times S^2$ with fixed point set $S^0 \cup S^0$. $1 \times R_1$ and $C \times K$ are the standard involutions with fixed point set $S^1 \cup S^1$. Now let $S^1 \times S^2$ be viewed as the identification space obtained from $[-1, 1] \times S^2$ by identifying $(-1, x)$ with $(1, R_1(x))$. The standard involutions $H$ and $L$ with fixed point set $S^1$ are defined by

$$H([t, x]) = [t, R_2(x)],$$
$$L([t, x]) = \begin{cases} [1-t, A(x)], & \text{if } 0 \le t \le 1, \\ [-1-t, K(x)], & \text{if } -1 \le t < 0. \end{cases}$$

Thus, we have

**Theorem A.** *If $F$ has dimension zero then $T$ is conjugate to the involution $C \times R_1$.*

**Theorem B.** *If $F$ has dimension one then $T$ is conjugate to one of the involutions $1 \times R_1, C \times K, H$, or $L$.*

Consequently, any PL involution of $S^1 \times S^2$ is conjugate to one of the above thirteen involutions. Using Theorem B, we prove in §4 the uniqueness of PL involutions of the solid torus with 1-dimensional fixed point set.

For fixed point free cyclic actions, we prove the following result in §5.

**Theorem C.** *$S^1 \times S^2$ admits just four free $Z_{2k}$-actions and just one free $Z_{2k+1}$-action for each positive integer $k$.*

Examples of closed, irreducible 3-manifolds admitting two-sided embeddings of $P^2$ (distinct from $P^2 \times S^1$) have recently been discovered by Row [9] and Jaco [4]. In §6 we describe a general construction that produces all such examples. This description should prove useful in the investigation of these spaces. The next theorem is an immediate consequence. If a closed 3-manifold $M$ contains $k$ nonparallel, disjoint, two-sided projective planes (and no more), the union of which does not separate $M$, then its orientable double cover is a closed 3-manifold $\widetilde{M}$ with $k$ handles attached.

**Theorem D.** *$M$ is irreducible if and only if $\widetilde{M}$ is irreducible.*

We work in the PL category exclusively. $S^n$, $P^n$, and $R$ denote the $n$-sphere, real projective $n$-space, and the real line, respectively. A 3-manifold $M$ is *irreducible* if every 2-sphere in $M$ is the boundary of a cell in $M$. $M$ is $P^2$-*irreducible* if $M$ is irreducible and admits no two-sided embeddings of $P^2$. A surface $F$ in $M$ is *properly embedded* in $M$ if $F \cap \partial M = \partial F$ ($\partial$ denotes the boundary). $F$ is *two-sided* if there is a neighborhood of $F$ in $M$ of the form $F \times [-1, 1]$ with $F = F \times 0$ and $F \times [-1, 1] \cap \partial M = \partial F \times [-1, 1]$. We say that two surfaces $F$ and $G$ are *parallel* in $M$ if and only if there exists an embedding of $F \times [-1, 1]$ in $M$ such that $F = F \times \{-1\}$ and $G = F \times \{1\}$. Let $\Phi$ be a homeomorphism of the surface $F$. $F \times [0, 1]/\Phi$ denotes the identification space resulting from $F \times [0, 1]$ by identifying $(x, 0)$ with $(\Phi(x), 1)$, for each $x \in F$.

Let $F$ be a two-sided surface in $M$. The manifold $M'$ obtained by *splitting* $M$ at $F$ is the manifold whose boundary contains two copies, $F^+$ and $F^-$, of $F$ such that the identification of $F^+$ and $F^-$ defines a natural projection $f$: $(M', F^+ \cup F^-) \to (M, F)$ with $f|M' - (F^+ \cup F^-)$ a homeomorphism onto $M - F$. Note that $M'$ is homeomorphic to $M - (F \times (-1, 1))$.

I am grateful to the referee for correcting an oversight in the proof of Theorem B.

2. **Fundamental lemmas.** In this section we prove two lemmas on which the results of this paper are founded. Given an involution on $S^1 \times S^2$, we prove the existence of appropriate 2-spheres along which we can equivariantly split $S^1 \times S^2$. This operation enables us to capitalize on the results of Livesay [6], [7], Smith [10], and Waldhausen [13] about involutions on the 3-sphere. Lemma 1 is used again in §5. $Z_k$ denotes the cyclic group of order $k$.

**Lemma 1.** *Let $M$ be a compact 3-manifold admitting an effective $Z_k$-action ($k = 2$ or 3) with fixed point set $F$. Let $T$ be a homeomorphism on $M$ generating the given $Z_k$-action. If $M$ contains a 2-sphere disjoint from $F$ that does not bound a 3-cell, then there exists a 2-sphere $S$ in $M$ that does not bound a 3-cell such that $S \cap F = \emptyset$ and either $TS = S$ (and $k = 2$) or $TS \cap S = \emptyset$.*

**Proof.** Let $p$: $M \to M^*$ be the projection onto the orbit space of the $Z_k$-action. A 2-sphere $S$ in $M - F$ is in general position with respect to $p$ provided that the singularities of $p(S)$ consist of only double curves and triple points. Let $\Sigma$ denote the set of all 2-spheres in $M - F$ that do not bound 3-cells in $M$ and that are in general position with respect to $p$. $\Sigma$ is not void since any 2-sphere in $M$ can be deformed by arbitrarily small isotopies of $M$ to obtain one in $M - F$ that is in general position with respect to $p$.

We define the complexity of a 2-sphere $S$ in $\Sigma$ to be the ordered pair $(t, d)$, where $t$ denotes the number of triple points of $p(S)$ and $d$ denotes the number of double curves of $p(S)$. Of course when $k = 2$ we always have $t = 0$. Consider the complexities of the 2-spheres in $\Sigma$ in lexigraphical order and select some $S$ in $\Sigma$ with minimal complexity. If the complexity of $S$ is $(0, 0)$, then we are done. However, if the complexity of $S$ is not $(0, 0)$, then we attempt to find another member of $\Sigma$ with smaller complexity. But because of our choice of $S$ this is impossible. We find that we are in the case $k = 2$ and we can obtain a 2-sphere $S'$ that does not bound a 3-cell and for which $TS' = S'$. Either way, the proof is completed.

Suppose that the complexity of $S$ is not (0, 0). Choose a simple closed curve $\alpha$ in $S \cap TS$ such that $\alpha$ bounds a disk $D$ in $S$ with the properties that $D \cap TS = \alpha$ and each component of $D \cap T^{-1}(S)$ meets $\alpha$.

*Case* 1. $T\alpha \neq \alpha$. Let $E = T^{-1}(D)$. Then $E \cap S = T^{-1}(\alpha)$, which separates $S$ into two disks $E_1$ and $E_2$ having $\alpha$ as their common boundary. At least one of the 2-spheres $E \cup E_1$ or $E \cup E_2$ does not bound a 3-cell in $M$, since $S$ does not bound one. Call this 2-sphere $S'$. We deform $S'$ slightly to put it in general position with respect to $p$ and to eliminate $\alpha$ as a double curve as follows: Let $E'$ be a disk close to and parallel to $E$ such that $\mathrm{Bd}(E') \cup \alpha$ bounds an annulus $A$ on $S'$ and $E'$ intersects $TS$ in the same manner as $E$ does. Take $S'' = E' \cup (S' - (A \cup E))$ as our new 2-sphere in $\Sigma$. We have removed $p(\alpha)$ as a double curve. If $p(\alpha)$ contained no triple points of $S$ then we have reduced the number of double curves and not increased the number of triple points. If $p(\alpha)$ does contain triple points, notice that $p(\alpha)$ contains all those triple points lying in $p(D)$. The number of double curves may be increased in this case, but we add no new triple points and remove all those triple points in $p(\alpha)$. In either event, the complexity of $S''$ is less than that of $S$. So this case is impossible.

*Case* 2. $T\alpha = \alpha$. Since we do not have triple curves, $k = 2$. Let $E = T(D)$. As before, $\alpha$ separates $S$ into two disks $D = E_1$ and $E_2$ having $\alpha$ as their common boundary. If $E \cup E_2$ did not bound a 3-cell, then by treating this 2-sphere as we did $S'$ in Case 1 we would obtain a member of $\Sigma$ of less complexity than $S$. Thus $E \cup E_2$ bounds a 3-cell. Consider the 2-sphere $S'' = D \cup E$. $S''$ does not bound a 3-cell and $TS'' = S''$.

The following corollary is proved by a routine application of Lemma 1.

**Corollary 1.** *Let $(M, p)$ be a regular triple-covering space of an irreducible ($P^2$-irreducible), compact 3-manifold. Then $M$ is also irreducible ($P^2$-irreducible).*

**Lemma 2.** *Let $M$ be a compact 3-manifold admitting an involution $T: M \to M$ with fixed point set $F$ homeomorphic to a closed 1-manifold (perhaps not connected). Suppose that $F$ intersects every 2-sphere in $M$ that does not bound a 3-cell. If $M$ is not irreducible, then there exists a 2-sphere $S$ in $M$ not bounding a 3-cell such that $TS = S$ and $S$ is in general position with respect to $F$.*

**Proof.** We say that a 2-sphere $S$ in $M$ is in $T$-general position modulo $F$ if $S$ and $TS$ are both in general position with respect to $F$ and if $S - F$ and $TS - F$ are in general position. Hence, we have four possible types of intersection curves or points in $S \cap TS$: (a) a simple closed curve in $S - F$, (b) a simple closed curve in $S$ that meets $F$ in a single point, (c) an arc in $S$ with just its endpoint in $F$, (d) an isolated point in $S \cap TS$ lying in $F$.

Let $\Sigma$ be the set of all 2-spheres in $M$ that do not bound 3-cells and are in $T$-general position modulo $F$. Clearly $\Sigma \neq \emptyset$. We define the complexity of $S \in \Sigma$ as $c(S) = (\alpha, \beta, \gamma, \delta)$, where $\alpha, \beta, \gamma$ denote the number of components in $(S \cap TS) - F$ that are of type a, b, c, respectively, and $\delta$ denotes the number of points in $S \cap TS$ of type d.

We choose some $S \in \Sigma$ with the smallest possible complexity (where the complexities are lexigraphically ordered). We attempt to reduce the complexity of $S$ by means similar to that used in Lemma 1. Because of our choice of $S$, there is only one possibility (Case 5) and we are able to construct an invariant 2-sphere not bounding a 3-cell.

*Case* 1. There is an intersection curve $\lambda$ of type a that bounds a disk $D$ in $TS$ such that $\mathrm{Int}(D) \cap S = \emptyset$. We use the same argument as in Lemma 1 to either remove $\lambda$ as an intersection curve or obtain an invariant sphere $D \cup TD$ containing no fixed points and not bounding a 3-cell. However, neither situation is compatible with our choice of $S$ and the conditions imposed on $F$.

*Case* 2. There is an intersection curve $\lambda$ of type b that bounds a disk $D$ in $TS$ such that $\mathrm{Int}(D) \cap S = \emptyset$. Let $\lambda \cap F = \{x\}$. $\lambda$ separates $S$ into two disks $E_1$ and $E_2$ having $\lambda$ as their common boundary. One of the 2-sphere $E_1 \cup D$ or $E_2 \cup D$ does not bound a 3-cell in $M$, say $S'$. We work with $S'$, deforming it slightly so as to put it in $T$-general position modulo $F$, and eliminate $\lambda$ as an intersection curve. Let $D'$ be a disk parallel and close to $D$ such that $\partial D' \cup \lambda = \{x\}$, $\partial D' \cup \lambda$ bounds a pinched annulus $A$ in $S'$, and the interior of the 3-cell bounded by $D \cup D' \cup A$ does not meet $S \cup TS$. Take $S'' = D' \cup (S' - A \cup D)$. Then $S'' \in \Sigma$ and $c(S'') < c(S)$ since we have not increased the number of double curves of type a and have decreased the number of type b (although perhaps increasing the number of points of type d). Thus Case 2 does not occur for our choice of $S$.

*Case* 3. There is a pair of intersection curves $\mu, \lambda$ of type c such that $\mu \neq T\lambda$ and $\mu \cup \lambda$ bounds a disk $D$ in $TS$ with $\mathrm{Int}(D) \cap S = \emptyset$. $\mu \cup \lambda$ separates $S$ into two disks $E_1, E_2$ with $\mu \cup \lambda = \partial E_1 = \partial E_2$. Select $S'$ to be one of the 2-spheres $D \cup E_1$, $D \cup E_2$ that does not bound a 3-cell. Let $D'$ be a disk parallel and close to $D$ such that $\partial D \cap \partial D' = \{x, y\}$ and $\partial D \cup \partial D'$ bounds an annulus $A$ in $S'$ "pinched" at both $x$ and $y$. Furthermore, we do not want the interior of the 3-cell bounded by $D \cup D' \cup A$ to meet $S \cup TS$. Take $S'' = D' \cup (S' - (A \cup D))$ as our new 2-sphere in $\Sigma$. Clearly $C(S'') < C(S)$ (again we may have introduced new points of type d). But as before, the existence of such a 2-sphere $S''$ is not consistent with our choice of $S$, so this case cannot occur either.

*Case* 4. There is a point $x$ of type d in $S \cap TS$. Let $N$ be an invariant 3-cell neighborhood of $x$ such that $N \cap F$ is an arc. Then $T|N$ is simply a rotation about this arc. We adjust $S$ and $TS$ slightly so that $S \cup TS$ is in general position with respect to $\partial N$. There are simple closed curves in $S \cap N$ and $TS \cap N$ that bound innermost disks (containing $x$) $R \subset S$ and $Q \subset TS$. $R \cup Q$ separates $N$ into three components $U, V, W$, where $R \subset \partial U$, $TU = V$ and $TW = W$. Clearly, $F \cap N \subset W$.

Let $D$ be a disk close to and parallel to $R$ such that $\mathrm{Int}(D) \subset U$ and $\partial D = \partial R$. Define $S'' = (S - R) \cup D$. The only difference between $S \cap TS$ and $S'' \cap TS''$ is that we have removed the point $\{x\}$. But as before, because of our choice of $S$, this case cannot appear.

*Case* 5. There is an intersection curve $\lambda$ of type c with $T\lambda \cup \lambda$ bounding a disk $D$ in $TS$ such that $\mathrm{Int}(D) \cap (S \cup TS) = \emptyset$. Suppose $S_1 = D \cup TD$ does not bound a 3-cell. Then we are finished, since $S_1$ is invariant under $T$. Indeed, we will show that it is the case that $S_1$ cannot bound a 3-cell. For, suppose that $S_1$ does bound a 3-cell. If we denote by $E$ the complement of $T(\mathrm{Int}(D))$ in $S$, then $S' = D \cup E$ does not bound a 3-cell when $S_1$ does. Take a disk $D'$ parallel and close to $D$ so that $\partial D \cap \partial D' = \{x, y\}$ and $\partial D \cup \partial D'$ bounds a doubly-pinched annulus $A$ in $S'$. Take $S'' = (S' - (D \cup A)) \cup D'$. Then $S''$ has fewer curves of type c than $S$ and no more of types a and b, which contradicts our choice of $S$.

Since at least one of the five cases must occur, and Case 5 is the only one consistent with our choice of $S$, we may conclude that there exists a $T$-invariant 2-sphere in $M$ that does bound a 3-cell.

3. **Involutions on $S^1 \times S^2$.** In this section we show that any PL involutions on $S^1 \times S^2$ with 0- or 1-dimensional fixed set is conjugate to one of the involutions $C \times R_1$, $1 \times R_1$, $C \times K$, $H$, or $L$. Let $T: S^1 \times S^2 \to S^1 \times S^2$ be an involution with fixed point set $F$. Choose a basepoint in $F$ and consider the isomorphism $T_*$ on the fundamental group $\Pi_1(S^1 \times S^2) \cong Z$ induced by $T$. Either $T_*(1) = 1$, in which case we write $T_* = I$, or $T_*(1) = -1$ and we write $T_* = -I$. Fremon [2] lists all possible fixed point sets $F$ of $T$:

| | $T$ preserves orientation | $T$ reserves orientation |
|---|---|---|
| $T_* = I$ | $S^1 \sqcup S^1$ | $S^1 \times S^1$ |
| | $S^1$ | Klein bottle |
| $T_* = -I$ | $S^1 \sqcup S^1$ | $S^2 \sqcup S^2$ |
| | $S^1$ | $S^0 \sqcup S^2$ |
| | | $S^0 \sqcup S^0$ |

We proceed by considering the possible 0- and 1-dimensional fixed point sets and apply Lemmas 1 and 2 to reduce the problem to considering in what way known involutions on the 3-sphere can induce involutions on $S^1 \times S^2$. We remark that it follows from [2], [5], and Theorem B that to each entry in the above table there corresponds a unique PL involution of $S^1 \times S^2$ (up to conjugation).

**Proof of Theorem A.** According to Fremon's table we must have $F \approx S^0 \cup S^0$. By Lemma 1 there exists a 2-sphere $S$ in $S^1 \times S^2$ that does not separate $S^1 \times S^2$ and such that either $TS = S$ or $TS \cap S = \emptyset$. Observe that we may assume $TS \cap S = \emptyset$. For if $TS = S$ and $T$ does not interchance the sides of $S$, we can split $S^1 \times S^2$ along $S$ to obtain $[0, 1] \times S^2$ with an involution $T'$ induced by $T$. Now cap the 2-sphere boundary components with 3-cells to obtain a 3-sphere and extend $T'$ to an involution on this 3-sphere. The fixed point set of this involution on the 3-sphere is $F$ plus two additional points. But such a fixed point set for an involution on the 3-sphere does not occur [10]. If $T$ does interchange the sides of $S$, we can replace $S$ by one of the boundary components of a product neighborhood of $S$.

Hence $S^1 \times S^2$ is separated into two components by $S \cup TS$, say $A$ and $B$, with $\bar{A} \approx \bar{B} \approx [0, 1] \times S^2$. Since $F \neq \emptyset$, $T\bar{A} = \bar{A}$ and $T\bar{B} = \bar{B}$. Cap the 2-sphere boundaries of $\bar{A}$ and $\bar{B}$ to obtain 3-sphere $\Sigma_A$ and $\Sigma_B$, respectively. $T$ induces involutions $T_A$, $T_B$ on $\Sigma_A$, $\Sigma_B$ with fixed point sets $F_A = F \cap A$, $F_B = F \cap B$, respectively. Again from [10], we observe that $F_A \approx F_B \approx S^0$.

$T_A$ and $T_B$ are equivalent to orthogonal involutions on $S^3$ with two fixed points [6]. Hence, we can find $T$-invariant 2-spheres $S_A \subset A$, $S_B \subset B$ that contain $F_A$, $F_B$, respectively. $S_A \cup S_B$ separates $S^1 \times S^2$ into two components $U$ and $V$, both homeomorphic to $[0, 1] \times S^2$. Let $J$ be an arc properly embedded in $U$ with one endpoint in $S_A$ and the other in $S_B$.

Now consider the corresponding situation for the standard action $C \times R$ on $S^1 \times S^2$: $C \times R$-invariant 2-spheres $S'_A$, $S'_B$ separating $S^1 \times S^2$ into two components $U'$ and $V'$, $S'_A$ and $S'_B$ each containing two fixed points, $C \times R(U') = V'$, and an arc $J'$ in $U'$ with one endpoint in each of $S'_A$ and $S'_B$.

Let $h_1: S_A \cup S_B \to S'_A \cup S'_B$ be a homeomorphism such that $T \mid S_A \cup S_B = h_1^{-1}(C \times R)h_1$. We may choose the arc $J'$ above so that its endpoints correspond to the endpoints of $J$ under $h_1$. Extend $h_1$ to a homeomorphism $h_2$: $U \to U'$ (this only involves extending $h_1$ over an open 3-cell). Now define $h$: $S^1 \times S^2 \to S^1 \times S^2$ by $h \mid U = h_2$ and $h \mid V = (C \times R)h_1(T \mid B)$. Then $h$ is a homeomorphism such that $hTh^{-1} = C \times R$. This proves Theorem A.

**Proof of Theorem B.**

*Case* 1. There exists a nonseparating 2-sphere disjoint from $F$. By Fremon's table we know that $T$ is orientation preserving. Applying Lemma 1 again, there is a nonseparating 2-sphere $S$ disjoint from $F$ such that either $TS = S$ or

$TS \cap S = \emptyset$. By an argument such as we used in the proof of Theorem A, we may assume that $TS \cap S = \emptyset$.

Now $S \cup TS$ separates $S^1 \times S^2$ into two components $A$, $B$, such that $TA = A$ and $TB = B$. Cap the 2-sphere boundaries with 3-cells to obtain 3-spheres $\Sigma_A$, $\Sigma_B$. $T$ induces involutions $T_A: \Sigma_A \to \Sigma_A$ and $T_B: \Sigma_B \to \Sigma_B$ with fixed point sets $F_A = F \cap A$ and $F_B = F \cap B$, respectively. There are two possibilities: (i) $F_A = \emptyset$, $F_B \approx S^1$; (ii) $F_A \approx F_B \approx S^1$. Observe that $T_A$ and $T_B$ are orientation preserving involutions on 3-spheres. Therefore, in (i), $T_A$ is equivalent to the antipodal map [7], $T_B$ is equivalent to a rotation of $S^3$ (with a circle of fixed points), and in (ii) both are equivalent to such a rotation of $S^3$ [13]. We can proceed just as we did at this stage in the proof of Theorem A. It follows that, in (i), $T$ is conjugate to the standard involution $L$ and, in (ii), $T$ is conjugate to the standard involution $C \times K$. This will complete the proof in Case 1 of Theorem B.

*Case* 2. Every nonseparating 2-sphere meets $F$. Consider an invariant 2-sphere $S$ that does not separate $S^1 \times S^2$. Split $S^1 \times S^2$ along $S$ to obtain $B$ homeomorphic to $S^2 \times [0, 1]$. $T$ induces an involution $T'$ on $B$ with (fixed point set $F$ split along $F \cap S$) that can be extended to the 3-sphere that results when one caps off the two boundary components of $B$ with 3-cells. By [13], $T'$ is conjugate to a rotation of the 3-sphere with fixed point set homeomorphic to the circle. Thus $F \cap S$ consists of just two points. To prove that $T$ is equivalent to $C \times K$ or $H$ we consider the two possibilities for $F$ separately.

*Subcase* a. $F$ is homeomorphic to $S^1 \cup S^1$. Each component of $F$ meets $S$ in one point. Let $J$ be an arc in $S$ with endpoints $F \cap S$ such that $T(J) \cap J = F \cap S$. There is a disk spanning $F \cap J$ with boundary identified along $J$ that forms a nonsingular annulus $A$ with boundary $F$. We may assume that there is a neighborhood $N$ of $F$ in $A$ such that $TN \cap N = F$. We may also choose $A$ so that $TA \cap A = F$. Simply take $A$ so that $(A - F) \cap (TA - F)$ consists of only simple closed curves and the fewest number possible. Then by a simple version of the argument used in Lemma 1, it follows that $(A - F) \cap (TA - F) = \emptyset$.

$TA \cup A$ is homeomorphic to $S^1 \times S^1$ and separates $S^1 \times S^2$ into two components (each homeomorphic to $D^2 \times S^1$) that are interchanged by $T$. A homeomorphism $h: S^1 \times S^2 \to S^1 \times S^2$ can be constructed in a fashion analogous to that used in Theorem A so that $T = h^{-1}(C \times K)h$. First define $h$ on $A \cup TA \cup S$, extend this homeomorphism over one of the 3-cell components of $S^1 \times S^2 - (A \cup TA \cup S)$, and then equivariantly over the other 3-cell component.

*Subcase* b. $F$ is homeomorphic to $S^1$. Let $p_T: S^1 \times S^2 \to X_T$ and $p_H: S^1 \times S^2 \to X_H$ denote the projections onto the orbit spaces of the involutions $T$ and $H$, respectively. Let $U$ and $V$ be invariant tubular neighborhoods of the fixed point sets of $T$ and $H$, respectively. Recall $T|B$ is equivalent to the restriction of a rotation of $S^3$. The spaces $S^1 \times S^2 - \mathrm{Int}(U)$, $X_T - \mathrm{Int}\, p_T(\mathrm{Int}(U))$, $S^1 \times S^2 - \mathrm{Int}(V)$, and $X_H - p_H(\mathrm{Int}(V))$ are homeomorphic to the orientable annulus bundle over $S^1$ with connected boundary, $A \times [0, 1]/\phi$. The fundamental group of each of these spaces is homeomorphic to that of the Klein bottle $K$, and can be presented by $K = (a, b: aba^{-1}b)$. Sewing the torus neighborhood $U$ (or $V$) back in adds a single relation. The only relation that can be added in this way so that the resulting group is infinite cyclic is the relation b. Hence there is essentially only one way in which the torus neighborhoods $U$ and $V$ can be sewn back in to give $S^1 \times S^2$. We apply this observation to extend a homeomorphism below.

Since $K$ has only one subgroup of index 2 isomorphic to $K$, the covering maps of $A \times I/\phi$ induced by $p_H$ and $p_T$ are equivalent. Hence, there exists a pair of homeomorphisms $\bar{g}: X_T - p_T(\mathrm{Int}(U)) \to X_H \times p_H(\mathrm{Int}(V))$ and $g: S^1 \times S^2 - \mathrm{Int}(U) \to S^1 \times S^2 - \mathrm{Int}(V)$ such that $p_H\bar{g} = gp_T | X_T - p_T(\mathrm{Int}(U))$. Taking note of the above observation, one can see that these homeomorphisms can easily be extended to homeomorphisms $\bar{h}: X_T \to X_H$ and $h: S^1 \times S^2 \to S^1 \times S^2$ such that $p_H\bar{h} = hp_T$. Therefore $T = h^{-1}Hh$. This completes the proof of Theorem B.

4. **Involutions on $D^2 \times S^1$.** It has been pointed out by Professor Kwun that Theorem B can be applied to give an affirmative answer to the following question: Is every PL involution of $D^2 \times S^1$ with 1-dimensional fixed point set equivalent to the rotation around the core $\{0\} \times S^1$? ($D^2$ denotes the unit disk in $R^2$.) Let $r: D^2 \to D^2$ denote the rotation through an angle of 180 degrees.

**Theorem E.** *Every PL involution of $D^2 \times S^1$ with 1-dimensional fixed point set is equivalent to $r \times 1$.*

**Proof.** Let $M = D^2 \times S^1$ and $h: M \to M$ be a PL involution with 1-dimensional fixed point set $F$. Let $M'$ be a disjoint copy of $M$ with a corresponding involution $h'$ with fixed point set $F'$. Consider the double of $M$, $2M$, obtained from $M$ and $M'$ by identifying them along their boundary by the identity map. Observe that $2M$ is homeomorphic to $S^1 \times S^2$ and that $h$ and $h'$ define an involution $\bar{h}$ on $S^1 \times S^2$. It is obvious that $F \subset \mathrm{Int}(M)$. Thus the fixed point set of $\bar{h}$ is $F \cup F'$. From Theorem B we see that $F \approx F' \approx S^1$ and that $\bar{h}$ is equivalent to either $1 \times R_1$ or $C \times K$.

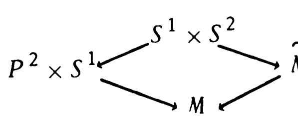
*Case* 1. $\bar{h}$ is equivalent to $1 \times R_1$. Let $N$ be a small invariant regular neighborhood of $F$ in $M$ (e.g., a second derived neighborhood of $F$). Consideration of the Mayer-Vietoris sequence of the pair $\{M - F, N\}$ reveals that $H_1(M - F; Z)$ has rank two. But $\pi_1(M - F)$ is a free abelian group, since $\pi_1(2M - (F \cup F')) \cong Z \times Z$ and $M - F$ is a retract of $2M - (F \cup F')$. Thus $\pi_1(M - \mathrm{Int}(N)) \cong Z \times Z$ and according to Stallings [16], $M - \mathrm{Int}(N)$ fibers over the circle with an annulus as fiber. $M - \mathrm{Int}(N)$ has two boundary components, so $M - \mathrm{Int}(N) \approx S^1 \times S^1 \times [0, 1]$. It is now easy to see that $h$ is equivalent to an involution on $M$ with fixed point set the core $\{0\} \times S^1$. $h \mid N$ is simply a rotation about the core. Let $X$ be the orbit space of $h \mid M - \mathrm{Int}(N)$. Then $X$ has two boundary components, one of which is homeomorphic to $S^1 \times S^1$ and is incompressible. The index of the subgroup of $\pi_1(X)$ carried by this boundary component must be the same as the index of $\pi_1(\partial N)$ in $\pi_1(M - \mathrm{Int}(N))$, namely one. Therefore $X$ is also homeomorphic to $S^1 \times S^1 \times [0, 1]$. One can now easily check that $h \mid M - \mathrm{Int}(N)$ is a rotation extending $h \mid \partial N$, and the theorem is proved in this case.

*Case* 2. $\bar{h}$ is equivalent to $C \times K$. Follow the same notation as in Case 1. However, $F$ is now null homotopic in $2M$ and hence in $M$. If we lift $h$ to an involution $\hat{h}$ on the universal covering space $(D^2 \times R, p)$ of $M$, we obtain an involution with fixed point set containing a component of $p^{-1}(F)$. But $p^{-1}(F)$ consists of an infinite number of components, each homeomorphic to $S^1$. Such an involution on $D^2 \times R$ as $\hat{h}$ does not exist. Thus Case 2 does not occur.

5. **Free cyclic actions on $S^1 \times S^2$.** In this section the main result of Tao in [11] classifying the free involutions on $S^1 \times S^2$ is extended to finite cyclic free actions. We use $N$ to denote the nonorientable 2-sphere bundle over the circle.

**Lemma 3.** $S^1 \times S^2$, $N$, *and* $P^2 \times S^1$ *admit only the obvious fixed point free involutions.*

**Proof.** We recall that $S^1 \times S^2$ is a double-covering only of $S^1 \times S^2$, $N$, $P^2 \times S^1$ and $P^3 \# P^3$ [11]. $N$ is a double-covering only of $P^2 \times S^1$ [12]. We also observe that $P^2 \times S^1$ only double-covers itself. Suppose $P^2 \times S^1 \to M$ is a double-covering. Let $\tilde{M} \to M$ be the orientable double-covering of $M$. By considering the commutative diagram of double-covering projections




one can see that $\tilde{M} \approx S^1 \times S^2$ and $M \approx P^2 \times S^1$.

**Proof of Theorem C.** *Let* $T: S^1 \times S^2 \to S^1 \times S^2$ be a homeomorphism generating the given free $Z_m$ action. Let $p: M \to M^*$ be the projection onto the orbit space $M^*$ of $T$. Consider the automorphism $T_*: H_1(S^1 \times S^2) \to H_1(S^1 \times S^2)$ induced by $T$. Either $T_*(1) = 1$ or $-1$, and in the latter case $T^2(1) = 1$. From the spectral sequence of the covering projection $p$ [8, p. 344] we obtain the following exact sequence:

$$0 \to K \to H_1(M^*) \to Z_m \to 0$$

where $K = Z$ if $T_*(1) = 1$ and $K = Z_2$ if $T_*(1) = -1$.

*Case* 1. $K = Z$. $H_1(M^*)$ is infinite. It follows that $M^*$ either contains a 2-sphere $F$ that does not bound a 3-cell or a two-sided incompressible surface $F$ [13]. Hence $p^{-1}(F)$ is a system of nonseparating 2-spheres in $S^1 \times S^2$. This means that $F$ is homeomorphic to $S^2$ or $P^2$. Each component of $S^1 \times S^2 - F$ is either double-covered or single-covered by a component of $S^1 \times S^2 - p^{-1}(F)$, each of which is homeomorphic to $S^2 \times (0, 1)$. Thus $M^*$ is homeomorphic to $S^1 \times S^2$, $N$, or $P^2 \times S^1$. If $M^*$ is nonorientable, $p$ can be factored through the orientable double-covering of $M^*$ and so $m$ must be even.

*Case* 2. $K = Z_2$. Then $H_1(M^*)$ is finite and $m$ is even. Consider the free cyclic action on $S^1 \times S^2$ generated by $T^2$ with orbit space $M'$. We can apply Case 1 to observe that $M'$ is homeomorphic to $S^1 \times S^2$, $P^2 \times S^1$, or $N$. Since $M'$ double-covers $M^*$, it follows by Lemma 3 that $M^*$ is homeomorphic to $S^1 \times S^2$, $P^2 \times S^1$, $N$, or $P^3 \# P^3$. However, only $P^3 \# P^3$ has finite first homology group. Therefore, $M^* \approx P^3 \# P^3$.

In both cases, notice that for each choice of $M^*$ there is only one choice for the covering projection $p$ (up to equivalence).

**6. Irreducible 3-manifolds containing two-sided $P^2$'s.** Let $T: M \to M$ be an involution on a compact, irreducible, orientable 3-manifold $M$ with fixed point set $\{x_1, \cdots, x_{2b+2k}\}$. $M$ need not be connected. We define the 3-manifold

$$N = [M, T; (x_1, x_2), \cdots, (x_{2k-1}, x_{2k}); x_{2k+1}, \cdots, x_{2k+2b}]$$

as follows:

Remove from $M$ a set of pairwise disjoint, invariant open 3-cell neighborhoods about each fixed point to obtain a 3-manifold $M'$ whose boundary has $(2k + 2b)$ 2-sphere boundary components $\{S_j\}$ ($S_j$ corresponding to the fixed point $x_j$). Let $T' = T \mid M'$. Denote the orbit space of $T'$ by $M^*$. $M^*$ has $2k + 2b$ $P^2$ boundary components $\{P_j\}$, where $P_j$ is covered by $S_j$. We construct the 3-manifold $N$ from $M^*$ by identifying $P_{2i-1}$ with $P_{2i}$ via a homeomorphism, for $i = 1, \cdots, k$. Define $\mathcal{P}$ to be the class of all connected 3-manifolds homeomorphic to some 3-manifold $N$ constructed in this manner.

The proof of Theorem D is a direct consequence of the following:

**Theorem F.** *$\mathcal{P}$ is the class of all compact, connected, irreducible 3-manifolds that admit two-sided embeddings of $P^2$.*

**Proof.** Suppose that $N$ is a compact, connected, irreducible 3-manifold containing two-sided projective planes. Let $\{P_1, \cdots, P_k\}$ be a maximal collection of pairwise disjoint, two-sided $P^2$'s in $N$ such that no pair are parallel and none are parallel to a boundary component of $N$ [15]. Let $\{P_{k+1}, \cdots, P_{k+2b}\}$ be the set of $P^2$ boundary components of $N$. Then $k + 2b > 0$. Let $\{U_j\}$ be a set of pairwise disjoint regular neighborhoods of the $P_j$ not in Bd($N$). Define $N_1 = \mathrm{Cl}(N - \bigcup_{j=1}^k U_j)$. $N_1$ has $2k + 2b$ $P^2$ boundary components. Let $(M_1, p)$ be the orientable double-covering of $N_1$ and let $T_1$ denote the covering transformation of $p$. Then $T_1$ is a free involution on $M_1$. Denote the 2-sphere boundary components of $M_1$ by $S_j$, indexed so that $S_{j+k} = p^{-1}(P_j)$ for $j = k+1, \cdots, k+2b$, and $S_{2j-1} \cup S_{2j} = p^{-1}(\mathrm{Bd}(U_j))$ for $j = 1, \cdots, k$. The 2-spheres $S_j$ are invariant under $T_1$. Form the 3-manifold $M$ by filling in all the 2-sphere boundary components $S_j$ with 3-cells and extend $T_1$ to an involution $T$ on $M$ with $2k + 2b$ fixed points—one in each of the 3-cells bounded by the $S_j$.

It follows that $N$ belongs to $\mathcal{P}$ once we prove that $M$ is irreducible. To prove this, suppose that $M$ contains a 2-sphere that does not bound a 3-cell and show that we are led to a contradiction. By Lemma 1 there is a 2-sphere $S$ in $M$ disjoint from the fixed points that does not bound a 3-cell and such that either $TS = S$ or $TS \cap S = \emptyset$.

*Case* 1. $TS \cap S = \emptyset$. In this case $p(S)$ is a 2-sphere in the irreducible 3-manifold $N_1$ and hence bounds a 3-cell. But this 3-cell must then lift to a 3-cell in $M$ bounded by $S$, which we have assumed does not bound a 3-cell.

*Case* 2. $TS = S$. Observe that $p(S)$ is a projective plane and that $S$ does not separate $M$. Since $N$ is irreducible, $T$ does not interchange the sides of $S$. For otherwise $N$ would have a $P^3$ summand. Hence $p(S)$ is a two-sided projective plane in $N_1$ which we may assume is disjoint from $\{P_j\}$. Thus $p(S)$ is parallel to some $P_j$ since $\{P_j\}$ is maximal. Consequently, $S$ is parallel to some $S_j$. But each $S_j$ bounds a 3-cell in $M$. Therefore, $S$ must also bound a 3-cell, which again is a contradiction. This proves that $M$ is irreducible.

Now let us suppose that $N$ is a 3-manifold belonging to $\mathcal{P}$. We show that $N$ is irreducible. We retain the notation introduced at the beginning of this section. Suppose that there exists a 2-sphere $S$ in $N$ that does not bound a 3-cell. We may assume that $S \cap P_j = \emptyset$. For if this were not the case, then we can find such a 2-sphere by the usual technique of picking an $S$ in general position with respect to the $P_j$'s and with a minimal intersection. If $S$ separates $N$, then one of the components of $M^*$, say $A^*$, is separated by $S$. Let $(M', q)$ be the orientable double cover of $M^*$. Then $q^{-1}(A^*) = A' \subset A$, where $A'$ and $A$ are components of $M'$ and $M$, respectively. One of the components of $A^* - S$, say $X$, lifts to two disjoint copies of $X$ in $A' - q^{-1}(S)$. Since $q^{-1}(X)$ is not connected, there are no projective planes in $\partial X$. Hence $q^{-1}(X)$ is homeomorphic to two disjoint open 3-cells, because $A$ is irreducible. But then $S$ would bound a 3-cell in $N$, in contradiction to our choice of $S$. If $S$ does not separate $N$, then $N$ would have a handle. In this situation, there would exist another 2-sphere disjoint from the $P_j$'s that separates $N$ but does not bound a 3-cell. But then again we contradict the irreducibility of $M$. Thus we have proved that every member of $\mathcal{P}$ is a compact, connected, irreducible 3-manifold that admits two-sided embeddings of $P^2$.

**Corollary 2.** *If $M \in \mathcal{P}$ and $\Pi_1(M)$ has a nontrivial center then $\Pi_1(M)$ is isomorphic to $Z \oplus Z_2$.*

**Proof.** The fundamental group of the orientable double-cover is isomorphic to either $Z$ or a nontrivial free product. In the latter case, all the elements in the center of $\Pi_1(M)$ have order 2. In either case, $\Pi_1(M)$ has a subgroup isomorphic to $Z \oplus Z_2$. It follows from [1] that $\Pi_1(M)$ is isomorphic to $Z \oplus Z_2$.

**Corollary 3.** *Let $M \in \mathcal{P}$ and suppose that $\Pi_1(M)$ has a nontrivial, normal, finitely generated subgroup $H$ not isomorphic to $Z$ such that the quotient group $\Pi_1(M)/H$ is infinite. Then $\Pi_1(M)$ is isomorphic to $Z \oplus Z_2$.*

**Proof.** It is proved in [3] that either $\Pi_1(M)$ is isomorphic to $Z \oplus Z_2$ or else the orientable double-covering is homeomorphic to some $A \# B$, where $B$ is a homotopy 3-sphere and $A$ is either $S^1 \times S^2$ or irreducible. Obviously $A = S^1 \times S^2$. Hence $\Pi_1(A \# B) \cong Z$ and $\Pi_1(M) \cong Z \oplus Z_2$.

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